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# A note on certain integrals of Bessel functions

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**Abstract.** In the classic book on Bessel functions by Watson, there appear to be minor errors in one particular integral of Bessel functions. This integral is given a series expansion which may be useful in certain physical problems. The asymptotic nature of this series is very briefly discussed.

## 1. Introduction

In the classic book on Bessel functions by Watson [1] there are two useful formulae, quoted without proof, credited to Lommel. They appear in Lommel's study of the Fresnel integral in the 1880s [2, in particular pp 600-1]. The first formula is

$$\int_0^z J_\nu(t)\,\mathrm{d}t = 2\sum_{n=0}^{\infty} J_{\nu+2n+1}(z) \tag{1a}$$

$$= \sum_{n=0}^{\infty} \frac{z^{n+1}J_{\nu+n}(z)}{(\nu+1)(\nu+3)\ldots(\nu+2n+1)}. \tag{1b}$$

The first equation (1*a*) is quite well known, found in standard references on higher transcendental functions [3]. The second equation (1*b*) is perhaps less well known.

The second formula is given as

$$\int_z^\infty J_\nu(t)\,\mathrm{d}t \sim \sum_{n=0}^{\infty} (\nu+1)(\nu+3)\ldots(\nu+2n-1)z^{-n}J_{\nu+n}(z) \tag{2}$$

where the symbol ($\sim$) presumably means 'asymptotically equal to', hence $|z|\gg 1$. The above formula is found in other references, for example [4], which cite Watson's book as their source. In certain physical problems, e.g. a two-dimensional electron gas or a harmonic oscillator chain, one encounters the following form for a time-dependent autocorrelation function $\mathscr{V}(t)$ [5]:

$$\mathscr{V}(t) = \sum_{n=0}^{\infty} (\nu+1)(\nu+3)\ldots(\nu+2n-1)\lambda^n t^{-n} J_{\nu+n}(t) \tag{3}$$

where $\lambda$ is some physical parameter and $t$ represents the time. Thus, for certain values of $\lambda$, such as $\lambda=1$, $\mathscr{V}(t)$ may be related to the formula due to Lommel.

The author believes that there are two minor errors in (2), most probably misprints. The correct form should have the order of the Bessel function as $\nu+n+1$ and a negative sign before the integral. In the original paper there appear no errors, but Lommel has given, as noted in Watson's book, only special cases for the above formula, namely $\nu=\frac{1}{2}$ and $-\frac{1}{2}$. Since the original source of this material is not likely to be easily accessible, and since the errors in Watson's book seem to have propagated into other references, it might be useful to provide a simple proof concerning this result. Our proof is valid for any $\nu$. It is slightly different from the proof due to Lommel, which is based on an integration factor for Bessel functions. We shall arrive at the same result using a recurrence relation for Bessel functions, somewhat reminiscent of our work on the method of recurrence relations for non-equilibrium statistical mechanics [6].

## 2. Proof of the statement

We begin with a well known recurrence relation for Bessel functions, written conveniently as

$$J_\nu(t) = J'_{\nu+1}(t) + (\nu+1)t^{-1}J_{\nu+1}(t) \tag{4}$$

where the prime means a derivative of the function with respect to its argument. Hence

$$\int_z^\infty J_\nu(t)\,dt = -J_{\nu+1}(z) + (\nu+1)\int_z^\infty t^{-1}J_{\nu+1}(t)\,dt \tag{5}$$

where we have assumed that $J_{\nu+1}(\infty)=0$†. Now $J_{\nu+1}(t)$ appearing under the integral sign in (5) may be replaced by two terms using (4), i.e.

$$\begin{aligned}\int_z^\infty t^{-1}J_{\nu+1}(t)\,dt &= \int_z^\infty t^{-1}J'_{\nu+2}(t)\,dt + (\nu+2)\int_z^\infty t^{-2}J_{\nu+2}(t)\,dt \\ &= -z^{-1}J_{\nu+2}(z) + (\nu+3)\int_z^\infty t^{-2}J_{\nu+2}(t)\,dt.\end{aligned} \tag{6}$$

Hence:

$$\int_z^\infty J_\nu(t)\,dt = -J_{\nu+1}(z) - (\nu+1)z^{-1}J_{\nu+2}(z) + (\nu+1)(\nu+3)\int_z^\infty t^{-2}J_{\nu+2}(t)\,dt. \tag{7}$$

The repetition of the above process yields the following general result:

$$\int_z^\infty J_\nu(t)\,dt = -\sum_{n=0}^{N} 1\times(\nu+1)(\nu+3)\ldots(\nu+2n-1)z^{-n}J_{\nu+n+1}(z) + R_N \tag{8}$$

where we have inserted unity to indicate the proper coefficient for the $n=0$ term. The remainder term is

$$R_N = (\nu+1)(\nu+3)\ldots(\nu+2N+1)\int_z^\infty t^{-N-1}J_{\nu+N+1}(t)\,dt. \tag{9}$$

If $\nu$ is a negative odd integer the remainer term vanishes and the series consists of a finite number of terms only, i.e. with $\mu=-\nu$:

$$\int_z^\infty J_\mu(t)\,dt = \sum_{n=0}^{N} 1\times(-\mu+1)(-\mu+3)\ldots(-\mu+2n-1)z^{-n}J_{-\mu+n+1}(z). \tag{10}$$

Note that for any integer $n$, $J_{-n}(z)=(-1)^nJ_n(z)$.

If $\nu$ is not a negative odd integer, the remainder $R_N$ term (9) may be finite. One can obtain an upper bound on $R_N$ as follows. For $|z|>1$:

$$|R_N| \leqslant (\nu+1)(\nu+3)\ldots(\nu+2N+1)\sup_{t\geqslant z}|J_{\nu+N+1}(t)|\int_z^\infty |t|^{-N-1}\,dt. \tag{11}$$

† If we set $\nu=0$ in (5) one gets a formula given by Luke [7].

**Table 1.** Equation (13) evaluated at $z=24$ for the first $N+1$ terms of the expansion. The $N=\infty$ entry means $J_0(z=24)$.

| $N$ | $-\sum_{n=0}^{N} n!2^n z^{-n} J_{n+2}(z)$ |
|---|---|
| 0 | −0.043 393 77 |
| 1 | −0.056 832 97 |
| 2 | −0.056 790 24 |
| 3 | −0.056 226 71 |
| 4 | −0.056 152 01 |
| 5 | −0.056 214 71 |
| 6 | −0.056 248 56 |
| 7 | −0.056 243 44 |
| 8 | −0.056 227 71 |
| 9 | −0.056 220 44 |
| 10 | −0.056 224 72 |
| 11 | −0.056 234 20 |
| 12 | −0.056 240 54 |
| 13 | −0.056 238 29 |
| 14 | −0.056 227 00 |
| 15 | −0.056 211 45 |
| 16 | −0.056 200 92 |
| 17 | −0.056 207 88 |
| ⋮ | |
| $\infty$ | −0.056 230 27 |

For a finite $N$ and for a sufficiently large value of $z$ one can thus estimate the upper bound. For $|z| \gg 1$, we shall therefore write (8) as

$$\int_z^\infty J_\nu(t)\,\mathrm{d}t \sim -\sum_{n=0}^{\infty} 1\times(\nu+1)(\nu+3)\ldots(\nu+2n-1)z^{-n}J_{\nu+n+1}(z). \tag{12}$$

In the appendix we have given an analysis for the remainder term in a related integral, which vanishes in the asymptotic limit.

We will briefly illustrate numerically the slow convergence in (12). Setting $\nu=1$, we have

$$J_0(z) = -\sum_{n=0}^{\infty} 2^n n!\,z^{-n}J_{n+2}(z). \tag{13}$$

In table 1 we have compared $J_0$ with a finite number of the leading terms of the right-hand side of (13) both evaluated at $z=24$. For a fixed value of $z$, the signs of $J_\nu$ oscillate slowly with $\nu$. Thus, the sum approaches $J_0(z=24)$ very gradually, not necessarily more closely with more terms.

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**Appendix. Proof of (1*b*)**

We shall show that the same idea may be used to obtain (1*b*). There is a second recurrence relation for Bessel functions, given as

$$J_\nu(t)=(t/\nu)J'_\nu(t)+(t/\nu)J_{\nu+1}(t). \tag{A1}$$

Hence:

$$\int_0^z J_\nu(t)\,\mathrm{d}t=\nu^{-1}\int_0^z tJ'_\nu(t)\,\mathrm{d}t+\nu^{-1}\int_0^z tJ_{\nu+1}(t)\,\mathrm{d}t. \tag{A2}$$

Integrating the first integral of the right-hand side of (A2) by parts, we obtain, if $\nu\neq -1$,

$$\int_0^z J_\nu(t)\,\mathrm{d}t=\frac{zJ_\nu(z)}{\nu+1}+\frac{1}{\nu+1}\int_0^z tJ_{\nu+1}(t)\,\mathrm{d}t. \tag{A3}$$

Now, as before, replace $J_{\nu+1}(t)$ under the integral sign on the right-hand side of (A3) by two terms using (A1). Thus, if $\nu\neq -3$,

$$\int_0^z tJ_{\nu+1}(t)\,\mathrm{d}t=\frac{1}{\nu+3}z^2J_{\nu+1}(z)+\frac{1}{\nu+3}\int_0^z t^2J_{\nu+2}(t)\,\mathrm{d}t. \tag{A4}$$

Combining (A3) and (A4), and repeating this process, we get

$$\int_0^z J_\nu(t)\,\mathrm{d}t=\sum_{n=0}^{N}\frac{z^{n+1}J_{\nu+n}(z)}{(\nu+1)(\nu+3)\ldots(\nu+2n+1)}+Q_N \tag{A5}$$

where

$$Q_N=\frac{1}{(\nu+1)(\nu+3)\ldots(\nu+2N+1)}\int_0^z t^{N+1}J_{\nu+N+1}(t)\,\mathrm{d}t \tag{A6}$$

provided that $\nu$ is not a negative odd integer.

An upper bound on the remainder term $Q_N$ can be determined as follows. Since $|J_\nu(t)|\leqslant 1$ for $\nu\geqslant 0$:

$$\begin{aligned}|Q_N|&\leqslant\frac{1}{(\nu+1)(\nu+3)\ldots(\nu+2N+1)}\int_0^z \mathrm{d}t\,t^{N+1}\\&=\frac{1}{(\nu+1)(\nu+3)\ldots(\nu+2N+1)}\,\frac{z^N}{N}.\end{aligned} \tag{A7}$$

Hence, if $|z|<1$, $Q_N\to 0$ as $N\to\infty$. Thus, unlike the large-$z$ expansion, one has an exact expansion for $|z|<1$:

$$\int_0^z J_\nu(t)\,\mathrm{d}t=\sum_{n=0}^{\infty}\frac{z^{n+1}J_{\nu+n}(z)}{(\nu+1)(\nu+3)\ldots(\nu+2n+1)} \tag{A8}$$

provided that $\nu$ is not a negative odd integer.

## References

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